

# Perfect Matchings in Claw-free Cubic Graphs

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## Abstract

Lovász and Plummer conjectured that there exists a fixed positive constant  $c$  such that every cubic  $n$ -vertex graph with no cutedge has at least  $2^{cn}$  perfect matchings. Their conjecture has been verified for bipartite graphs by Voorhoeve and planar graphs by Chudnovsky and Seymour. We prove that every claw-free cubic  $n$ -vertex graph with no cutedge has more than  $2^{n/12}$  perfect matchings, thus verifying the conjecture for claw-free graphs.

## 1 Introduction

A graph is *claw-free* if it has no induced subgraph isomorphic to  $K_{1,3}$ . A graph is *cubic* if every vertex has exactly three incident edges. A well-known classical theorem of Petersen [9] states that every cubic graph with no cutedge has a perfect matching. Sumner [10] and Las Vergnas [6] independently showed that every connected claw-free graph with even number of vertices has a perfect matching. Both theorems imply that every claw-free cubic graph with no cutedge has at least one perfect matching.

In 1970s, Lovász and Plummer conjectured that every cubic graph with no cutedge has exponentially many perfect matchings; see [7, Conjecture 8.1.8]. The best lower bound has been obtained by Esperet, Kardoš, and Král' [5]. They showed that the number of perfect matchings in a sufficiently large cubic graph with no cutedge always exceeds any fixed linear function in the number of vertices.

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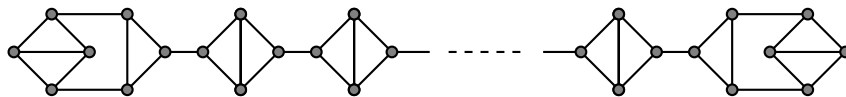


Figure 1: Claw-free cubic graphs with only 9 perfect matchings

So far the conjecture is known to be true for bipartite graphs and planar graphs. For bipartite graphs, Voorhoeve [11] proved that every *bipartite* cubic  $n$ -vertex graph has at least  $6(4/3)^{n/2-3}$  perfect matchings. Recently, Chudnovsky and Seymour [2] proved that every *planar* cubic  $n$ -vertex graph with no cutedge has at least  $2^{n/655978752}$  perfect matchings.

We prove that every claw-free cubic  $n$ -vertex graph with no cutedge has more than

$$2^{n/12}$$

perfect matchings. The graph should not have any cutedge; in Figure 1, we provide an example of a claw-free cubic graph with only 9 perfect matchings.

Our approach is to use the structure of 2-edge-connected claw-free cubic graphs. The *cycle space*  $\mathcal{C}(H)$  of  $H$  is a collection of the edge-disjoint union of cycles of  $H$ . It is well known that  $\mathcal{C}(H)$  forms a vector space over  $GF(2)$  and

$$\dim \mathcal{C}(H) = |E(H)| - |V(H)| + 1$$

if  $H$  is connected, see Diestel [3]. Roughly speaking, almost all 2-edge-connected claw-free cubic graph  $G$  can be built from a 2-edge-connected cubic multigraph  $H$  by certain operations so that every member of  $\mathcal{C}(H)$  can be extended to 2-factors of  $G$ . We will have two cases to consider; either  $H$  is big or small. If  $H$  is big, then  $\mathcal{C}(H)$  is big enough to prove that  $G$  has many 2-factors. If  $H$  is small, then we find a 2-factor of  $H$  using many of the specified edges of  $H$  so that when transforming this 2-factor of  $H$  to that of  $G$ , each of those edges of  $H$  has many ways to make 2-factors of  $G$ .

## 2 Structure of 2-edge-connected claw-free cubic graphs

Graphs in this paper have no parallel edges and no loops, and multigraphs can have parallel edges and loops. We assume that a loop is counted twice when measuring a degree of a vertex in a multigraph. Every 2-edge-connected cubic multigraph can not have loops because if it has a loop, then it must have a cutedge.

We describe the structure of claw-free cubic graphs given by Palmer et al. [8]. A *triangle* of a graph is a set of three pairwise adjacent vertices. *Replacing a vertex*

$v$  with a triangle in cubic graph is to replace  $v$  with three vertices  $v_1, v_2, v_3$  forming a triangle so that if  $e_1, e_2, e_3$  are three edges incident with  $v$ , then  $e_1, e_2, e_3$  will be incident with  $v_1, v_2, v_3$  respectively.

Every vertex in a claw-free cubic graph is in 1, 2, or 3 triangles. If a vertex is in 3 triangles, then the component containing the vertex is isomorphic to  $K_4$ . If a vertex is in exactly 2 triangles, then it is in an induced subgraph isomorphic to  $K_4 \setminus e$  for some edge  $e$  of  $K_4$ . Such an induced subgraph is called a *diamond*. It is clear that no two distinct diamonds intersect.

A *string of diamonds* is a maximal sequence  $D_1, D_2, \dots, D_k$  of diamonds in which, for each  $i \in \{1, 2, \dots, k-1\}$ ,  $D_i$  has a vertex adjacent to a vertex in  $D_{i+1}$ . A string of diamonds has exactly two vertices of degree 2, which are called the *head* and the *tail* of the string. Replacing an edge  $e = uv$  with a string of diamonds with the head  $x$  and the tail  $y$  is to remove  $e$  and add edges  $ux$  and  $vy$ .

A connected claw-free cubic graph in which every vertex is in a diamond is called a *ring of diamonds*. We require that a ring of diamonds contains at least 2 diamonds. It is now straightforward to describe the structure of 2-edge-connected claw-free cubic graphs as follows.

**Proposition 1.** *A graph  $G$  is 2-edge-connected claw-free cubic if and only if either*

- (i)  *$G$  is isomorphic to  $K_4$ ,*
- (ii)  *$G$  is a ring of diamonds, or*
- (iii)  *$G$  can be built from a 2-edge-connected cubic multigraph  $H$  by replacing some edges of  $H$  with strings of diamonds and replacing each vertex of  $H$  with a triangle.*

*Proof.* Let us first prove the “if” direction. It is easy to see that  $G$  is 2-edge-connected cubic and has no loops or parallel edges. If  $G$  is built as in (iii), then clearly  $G$  has neither loops nor parallel edges, and every vertex of  $G$  is in a triangle and therefore  $G$  is claw-free. Note that since  $H$  is 2-edge-connected,  $H$  can not have loops.

To prove the “only if” direction, let us assume that  $G$  is a 2-edge-connected claw-free cubic graph. We may assume that  $G$  is not isomorphic to  $K_4$  or a ring of diamonds. We claim that  $G$  can be built from a 2-edge-connected cubic multigraph as in (iii). Suppose that  $G$  is a counter example with the minimum number of vertices.

If  $G$  has no diamonds, then every vertex of  $G$  is in exactly one triangle and therefore  $V(G)$  can be partitioned into disjoint triangles. By contracting each triangle, we obtain a 2-edge-connected cubic multigraph  $H$ .

So  $G$  must have a string of diamonds. Let  $D$  be the set of vertices in the string of diamonds. Since  $G$  is cubic,  $G$  has two vertices not in  $D$ , say  $u$  and  $v$ , adjacent

to  $D$ . If  $u = v$ , then because the degree of  $u$  is 3,  $u$  must have another incident edge  $e$  but  $e$  will be a cutedge of  $G$ . Thus  $u \neq v$ .

If  $u$  and  $v$  are adjacent in  $G$ , then  $u$  and  $v$  must have a common neighbor  $x$ , because otherwise  $G$  will have an induced subgraph isomorphic to  $K_{1,3}$ . However one of the edges incident with  $x$  will be a cutedge of  $G$ , a contradiction.

Thus  $u$  and  $v$  are nonadjacent in  $G$ . Let  $G' = (G \setminus D) + uv$ , that is obtained from  $G$  by deleting  $D$  and adding an edge  $uv$ . Then  $G'$  has no parallel edges or loops and moreover  $G'$  is 2-edge-connected claw-free cubic. Since  $G$  has a vertex not in a diamond, so does  $G'$  and therefore  $G'$  can be built from a 2-edge-connected cubic multigraph  $H$  by replacing some edges with strings of diamonds and replacing each vertex of  $H$  with a triangle. Since  $D$  is chosen maximally,  $u$  and  $v$  are not in diamonds and therefore  $H$  has the edge  $uv$ . So we can obtain  $G$  from  $H$  by doing all replacements to obtain  $G'$  and then replacing the edge  $uv$  with a string of diamonds. This completes the proof.  $\square$

We remark that Proposition 1 can be seen as a corollary of the structure theorem of quasi-line graphs by Chudnovsky and Seymour [1]. A graph is a *quasi-line* graph if the neighborhood of each vertex is expressible as the union of two cliques. It is obvious that every claw-free cubic graph is a quasi-line graph. Chudnovsky and Seymour [1] proved that every connected quasi-line graph is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips. For 2-edge-connected claw-free cubic graphs, a fuzzy circular interval graph corresponds to a ring of diamonds and a composition of fuzzy linear interval strips corresponds to the construction (iii) of Proposition 1.

### 3 Main theorem

**Theorem 2.** *Every claw-free cubic  $n$ -vertex graph with no cutedge has more than  $2^{n/12}$  perfect matchings.*

*Proof.* Let  $G$  be a claw-free cubic  $n$ -vertex graph with no cutedge. We may assume that  $G$  is connected. If  $G$  is isomorphic to  $K_4$ , then the claim is clearly true. If  $G$  is a ring of diamonds, then  $G$  has  $2^{n/4} + 1$  perfect matchings. Thus we may assume that  $G$  is obtained from a 2-edge-connected cubic multigraph  $H$  by replacing some edges of  $H$  with strings of diamonds and replacing each vertex of  $H$  with a triangle.

Let  $k = |V(H)|$ . In other words,  $3k$  is the number of vertices not in a diamond of  $G$ .

Suppose that  $k \geq n/6$ . Since  $H$  has  $3k/2$  edges, the cycle space of  $H$  has dimension  $3k/2 - k + 1 = k/2 + 1$  and therefore  $|\mathcal{C}(H)| = 2^{k/2+1}$ . To obtain a 2-factor from  $C \in \mathcal{C}(H)$ , we transform  $C$  into a member  $C' \in \mathcal{C}(G)$  so that it meets all 3 vertices of  $G$  corresponding to  $v$  for each vertex  $v$  of  $H$  incident with

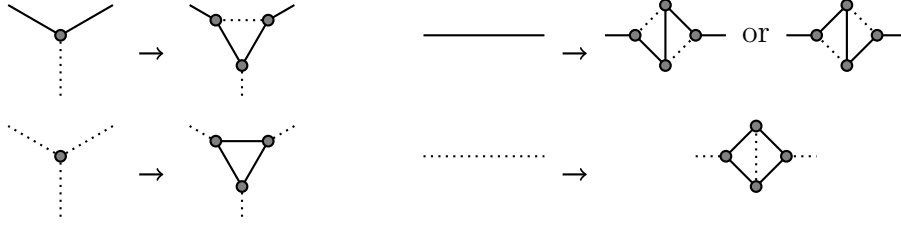


Figure 2: Transforming a member of  $\mathcal{C}(H)$  into a 2-factor of  $G$  (Solid edges represent edges in a member of  $\mathcal{C}(H)$  or a 2-factor of  $G$ .)

$C$  as well as it meets all the vertices in each diamond that corresponds to an edge in  $C$ . Then for each vertex  $w$  of  $G$  unused yet in  $C'$ , we add a cycle of length 3 or 4 depending on whether the vertex is in a diamond; see Figure 2. Then this is a 2-factor of  $G$  because it meets every vertex of  $G$ . Since the complement of the edge-set of a 2-factor is a perfect matching, we conclude that  $G$  has at least  $2^{k/2+1} \geq 2^{n/12+1}$  perfect matchings.

Now let us assume that  $k < n/6$ . We know that  $G$  has  $(n - 3k)/4$  diamonds. The *length* of an edge  $e$  of  $H$  is the number of diamonds in the string of diamonds replaced with  $e$ . (If the edge  $e$  is not replaced with a string of diamonds, then the length of  $e$  is 0.)

Edmonds' characterization of the perfect matching polytope [4] implies that there exist a positive integer  $t$  depending on  $H$  and a list of  $3t$  perfect matchings  $M_1, M_2, \dots, M_{3t}$  in  $H$  such that every edge of  $H$  is in exactly  $t$  of the perfect matchings. (In other words,  $H$  is fractionally 3-edge-colorable.) By taking complements, we have a list of  $3t$  2-factors of  $H$  such that each edge of  $H$  is in exactly  $2t$  of the 2-factors in the list. Since  $G$  has  $(n - 3k)/4$  diamonds, the sum of the length of all edges of  $H$  is  $(n - 3k)/4$ . Therefore there exists a 2-factor  $C$  of  $H$  whose length is at least  $\frac{n-3k}{4} \frac{2}{3} = (n - 3k)/6$ .

We claim that  $G$  has at least  $2^{(n-3k)/6}$  2-factors corresponding to  $C$ . For each diamond in the string replacing an edge  $e$  of  $C$ , there are two ways to route cycles of  $C$  through the diamond, see Figure 2. Since  $C$  passes through at least  $(n - 3k)/6$  diamonds,  $G$  has at least  $2^{(n-3k)/6}$  2-factors. Since  $k < n/6$ ,  $G$  has more than  $2^{n/12}$  2-factors. Thus  $G$  has more than  $2^{n/12}$  perfect matchings.  $\square$

We remark that every 3-edge-connected claw-free cubic  $n$ -vertex graph  $G$  has exactly  $2^{n/6+1}$  perfect matchings, unless  $G$  is isomorphic to  $K_4$ . That is because  $G$  has no diamonds and so, from the idea of the above proof, there is a one-to-one correspondence between the set of all 2-factors of  $G$  and the cycle space of a multigraph  $H$  obtained by contracting each triangle of  $G$ .

## References

- [1] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 153–171. Cambridge Univ. Press, Cambridge, 2005.
- [2] M. Chudnovsky and P. Seymour. Perfect matchings in planar cubic graphs. Submitted, 2008.
- [3] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.
- [4] J. Edmonds. Maximum matching and a polyhedron with 0, 1-vertices. *J. Res. Nat. Bur. Standards Sect. B*, 69B:125–130, 1965.
- [5] L. Esperet, F. Kardoš, and D. Král'. Cubic bridgeless graphs have more than a linear number of perfect matchings. Accepted to Eurocomb'09, 2009.
- [6] M. Las Vergnas. A note on matchings in graphs. *Cahiers Centre Études Recherche Opér.*, 17(2-3-4):257–260, 1975. Colloque sur la Théorie des Graphes (Paris, 1974).
- [7] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. *Annals of Discrete Mathematics*, 29.
- [8] E. M. Palmer, R. C. Read, and R. W. Robinson. Counting claw-free cubic graphs. *SIAM J. Discrete Math.*, 16(1):65–73 (electronic), 2002.
- [9] J. Petersen. Die Theorie der regulären graphs. *Acta Math.*, 15(1):193–220, 1891.
- [10] D. P. Sumner. Graphs with 1-factors. *Proc. Amer. Math. Soc.*, 42:8–12, 1974.
- [11] M. Voorhoeve. A lower bound for the permanents of certain  $(0, 1)$ -matrices. *Nederl. Akad. Wetensch. Indag. Math.*, 41(1):83–86, 1979.